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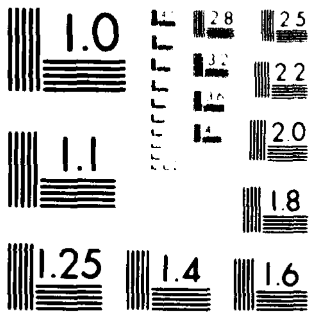
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NAVAL POSTGRADUATE SCHOOL

Monterey, California

**CALCULATION OF STATE PROBABILITIES
FOR A
STOCHASTIC LANCHESTER COMBAT MODEL**

by

L. Billard

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Lanchester (1914) presented his original combat model between two forces in a deterministic framework. Here, it is shown how the underlying state probabilities of a stochastic analogue of Lanchester's model can be calculated.

KEYWORDS: Lanchester combat process,
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1. INTRODUCTION

Lanchester (1914) presented a mathematical model describing the conflict situation of two forces in combat losing units due to attrition. This model was deterministic in nature and as, for example, in Gye and Lewis (1976) this approach is known to break down in certain cases most especially when the two sides are nearly or equally matched. In these circumstances, it is more imperative that the process be described from a stochastic outlook.

Therefore Billard (1979) considers how stochastic analogues of many deterministic Lanchester-type combat models can be formulated. Basically the processes can be represented as either a bivariate death process or a bivariate birth and death process. That paper includes a brief discussion of how the resultant Chapman-Kolmogorov (differential-difference) equations can be solved to provide expressions for the underlying state probabilities. The techniques employed are general in nature and can be used successfully on a very wide ranging array of model situations.

It is proposed here to demonstrate just how these techniques can be employed. This is done using the original Lanchester process, a process of great interest in its own right. First, however, the basic mathematical deterministic and stochastic models for the Lanchester process is reviewed briefly in Section 2. A discussion of the Severo (1969) recursion solution technique is presented in Section 3 as it pertains to the special nature of our problem. A partitioning scheme which further facilitates exploitation of the underlying structure is described in Section 4. Finally, in Section 5 the solution is developed.

2. THE LANCHESTER MODEL

Let $B(t)$ and $R(t)$ be the number of units (men, tanks, etc.) at time t in the Blue and Red force, respectively. Let particular values of these (integer valued) variables be b and r, respectively. Let $B(0) = B_0$ and $R(0) = R_0$, assumed to be known nonnegative quantities. The two forces are in combat with each other and lose units by attrition. In the original Lanchester (1914) model formulation, the force sizes at time t are modelled deterministically according to the equations

$$\frac{db}{dt} = -\gamma_1 r \tag{1}$$

and

$$\frac{dr}{dt} = -\gamma_2 b ,$$

where γ_1 and γ_2 are the attrition rates of the Red and Blue forces, respectively. For convenience, we rescale so that $\gamma_2 = 1$ and $\gamma_1 \equiv \lambda$ where now λ is the relative effectiveness of the Blue force to the Red force.

A stochastic model for this Lanchester model can be presented when we view the process as a bivariate pure death process. Thus $B(t)$ and $R(t)$ are now random variables. Suppose

$$p_{b,r}(t) = \Pr\{B(t) = b,\ R(t) = r\}.$$

Then, the corresponding Chapman-Kolmogorov (differential-difference) equations are

$$\frac{d}{dt} p_{b,r}(t) = -(\lambda r + b)\, p_{b,r}(t) + \lambda r p_{b+1,r}(t) + b p_{b,r+1}(t) \quad , \qquad (2a)$$

for $(b,r) \in A = \{(b,r): 0 \leq b \leq B_0,\ 0 \leq r \leq R_0\}$. For boundary values of A, certain obvious adjustments are necessary in (2a). Thus, when $b = B_0$, that is, for points (B_0,r) in A, $r \neq 0$ and $r \neq R_0$ the Chapman-Kolmogorov equations are

$$\frac{d}{dt} p_{B_0,r}(t) = -(\lambda r + B_0) p_{B_0,r}(t) + B_0 p_{B_0,r+1}(t) \ ; \qquad (2b)$$

when $r = R_0$, that is, for points (b,R_0) in A, $b \neq B_0$,

$$\frac{d}{dt} p_{r,R_0}(t) = -(\lambda R_0 + b)\, p_{b,R_0}(t) + \lambda R_0 p_{b+1,R_0}(t) \ ; \qquad (2c)$$

when $r = 0$, that is, for points $(b,0)$ in A,

$$\frac{d}{dt} p_{b,0}(t) = b p_{b,1}(t) \ ; \qquad (2d)$$

when $b = 0$, that is, for points $(0,b)$ in A,

$$\frac{d}{dt} p_{0,r}(t) = \lambda r p_{1,r}(t) \ ; \qquad (2e)$$

and when $(b,r) = (B_0,R_0)$,

$$\frac{d}{dt} p_{B_0,R_0}(t) = -(\lambda R_0 + B_0)\, p_{B_0,R_0}(t) \ . \qquad (2f)$$

These equations (2) can be solved by adopting the technique of Severo (1969). The first step then is to identify each point (b,r) in A by a counting coordinate k given by

$$k = k(b,r) = (B_0 + 1)\ (R_0 + 1) - b(R_0 + 1) - r\ .$$

Then if we set

$$y_k(t) = p_{b,r}(t)\ , \tag{3}$$

we may rewrite (2) in terms of the new coordinate k. We suffice by writing (2a) only, the other parts of (2) following similarly. Thus, equation (2a) becomes

$$\frac{d}{dt}\, y_k(t) = -(\lambda r + b)\ y_k(t) + \lambda r\ y_{k-R_0-1}(t) + by_{k-1}(t)\ . \tag{4}$$

Or, in matrix terms

$$\frac{d}{dt}\, \underline{y}(t) = \underline{By}(t)\ . \tag{5}$$

Clearly, from (4), the matrix of coefficients $\underline{B}$ is lower triangular. Therefore we can use Severo's recursive result to give us

$$\underline{y}(t) = \underline{Ce}(t) \tag{6}$$

where $\underline{e}(t)$ is the $(B_0 + 1)(R_0 + 1) \times 1$ vector with elements $\exp(b_k t)$ with b_k being the kth diagonal element of $\underline{B}$. The elements $c(i,j)$, $i, j = 1, \ldots, (B_0 + 1)(R_0 + 1)$, of the matrix

$\underline{C}$ can be found from Severo (1969, Theorem 1). That is,

$$c(i,j) = \begin{cases} a_1 , & i = j = 1, \\ a_i - \sum_{u=1}^{i-1} c(i,u) , & i = j \geq 2, \\ \underline{b}'(i,i-1)\ \underline{C}(i-1,j)\ \underline{h}(j,i-1) , & i > j, \\ 0 , & i < j, \end{cases} \tag{7}$$

where

$$c(i,j) = c_0(i, j) + tc_1(i,j) + \cdots + t^{i-j}c_{i-j}(i,j);$$

$$\underline{b}'(i,j) = (b_{i1}, \ldots , b_{ij}) , \qquad i \geq j , \tag{8}$$

with b_{ij} the (ij)th element of $\underline{B}$;

$$\underline{h}'(j,i) = (\delta_0(b_j - b_i),\ldots, \delta_{i-j-1}(b_j-b_i) , \qquad i > j , \tag{9}$$

with

$$\delta_i(y) = \begin{cases} (i!/y^{i+1}) \sum_{j=0}^{i} (-1)^{i-j} (yt)^j/j! , & y \neq 0 , \\ t^{i+1}/(i+1), & y = 0; \end{cases}$$

$$(\underline{C}(i,j))_{rs} = \begin{cases} c_{s-1}(r,j) , & r \geq j + s - 1, \\ 0 & \text{otherwise} , \end{cases} \tag{10}$$

for $r = 1,\ldots, i$ and $s = 1,\ldots, i-j+1$; and $a_k = y_k(0)$, $k = 1,\ldots, (B_0+1)(R_0+1)$.

3. THE SOLUTION MATRIX $\underline{C}$

The expressions for finding C given in the previous section are the general ones given by Severo. However, a close examination of our particular problem shows that some interesting simplifications can be obtained.

Typically, $\underline{a} = (1,0,\ldots,0)$. This corresponds to $B(0) = B_0$ and $R(0) = R_0$ with probability one. We shall assume this holds in the rest of this paper. The adjustment for the general $\underline{a}$ case is trivial.

We notice that $\underline{b}(i,i-1)$ is simply the ith row of the subdiagonal matrix of $\underline{B}$. From (2) (or (5)), in our case this vector has at most two "non-zero" elements corresponding to the coefficients of $y_{k-1}(r)$ and $y_{k-R_0-1}(t)$. Note that in some instances these values may "happen to be" zero. All other elements of $\underline{b}(i,i-1)$ are zero. We exploit this fact in the actual determination of the elements of $\underline{C}$.

A further simplification occurs when the coefficients of $\underline{B}$ are time independent as is the situation for our Lanchester model. We first note that the recursiveness in the solution enters through the $\underline{C}_{rs}(i-1,j)$ matrix. We also observe that in general we can express obtained elements $c(i,j)$ as a polynomial in t, viz.,

$$c(i,j) = c_0(i,j) + tc_1(i,j) + \cdots + t^{i-j}c_{i-j}(i,j) .$$

Thus, the elements in the first column of $\underline{C}_{rs}(i-1,j)$ are the appropriate coefficients of t^0 in the expansion of the $c(i,j)$'s, the elements in the second column are the appropriate coefficients of t^1 in the expansion of the $c(i,j)$'s, and so on. Now, in many problems of practical interest, and in particular in our present model, the elements $c(i,j)$ are time independent. That is, all elements in columns other than the first are zero. Hence, in $\underline{h}(i-1,j)$ we need only determine the first element $\delta_0(b_j - b_i)$. As a final comment, if $b_j = b_i$, then $\delta_0(0)$ is a function of t. However, typically this situation arises when we are in a row i whose $\underline{b}(i,i-1)$ $\underline{C}_{rs}(i-1,j)$ elements are all zero. Thus, the independence of $c(i,j)$ on t remains intact.

By exploiting these features, we can determine more readily the elements of $\underline{C}$. Before doing this (in Section 5), we shall present a partitioned form of $\underline{B}$ and hence $\underline{C}$ which allows us to observe much of this underlying structure at a glance.

4. A PARTITIONING SCHEME

First we note that the counting coordinate k orders the points (b,r) in A as follows: (B_0,R_0), $(B_0,R_0-1),\ldots,(B_0,0)$; $(B_0-1,R_0),\ldots,$ $(B_0-1,0);\ldots;$ $(0,R_0),\ldots,(0,0)$. Thus we partition these into groups whose b coordinate is common. That is, we may write

$$\underline{B} = (\underline{b}_{uv}), \qquad u, v = 1,\ldots, B_0+1,$$

and the "elements," or submatrices, $\underline{b}_{uv}$ have elements

$$\underline{b}_{uv} = (b_{uv}(p,q)), \qquad p, q = 1,\ldots, R_0+1 .$$

This is illustrated in Table 1 for the case $(B_0,R_0) = (4,2)$. Thus, the $k = k(b,r)$<u>th</u> row of $\underline{B}$ corresponds to the $p = (R_0-r+1)$<u>th</u> row of the $u = (B_0-b+1)$<u>th</u> submatrix of $\underline{B}$; similarly for the columns of $\underline{B}$.

To insert the appropriate coefficients in $\underline{B}$, we refer to equation (2) (or (4)). It is immediately apparent that the coefficients of $p_{b,r}(t)$ (or $y_k(t)$) constitute the diagonal elements of $\underline{B}$; the coefficients of $p_{b,r+1}(t)$ (or $y_{k-1}(t)$) consistute the off-diagonal elements of the diagonal submatrices; the coefficients of $p_{b+1,r}(t)$ (or $y_{k-R_0-1}(t)$) constitute the diagonal elements of the off-diagonal submatrices; and all other elements are zero. If we assert that off-diagonal elements of a submatrix cannot go "outside" that submatrix, then the adjustments to equation (2a) which produced the remaining equations of (2) when dealing with boundary values of (b,r) are automatically taken care of. Thus, for example, when $(B_0,R_0) = (4,2)$ the equation relating to the boundary point (2,2) does not have a term corresponding to $p_{2,3}(t)$ (or $y_{k-1}(t)$). If so, our $\underline{B}$ matrix would have an entry in the spot indicated by * in Table 1.

Thus, in our Lanchester model, $\underline{B}$ has elements

$$b_{uu}(p,p) = \begin{cases} -\{\lambda(R_0-p+1) + (B_0-u+1)\}, & u = 1,\ldots,B_0,\ p=1,\ldots,R_0, \\ 0, & u = B_0+1 \text{ and/or } p = R_0+1; \end{cases}$$

$$b_{uu}(p,p-1) = (B_0-u+1), \qquad u = 1,\ldots,B_0+1,\ p = 2,\ldots,R_0+1;$$

and

$$b_{u,u-1}(p,p) = \lambda(R_0-p+1), \qquad u = 2,\ldots,B_0+1,\ p = 1,\ldots,R_0+1.$$

Clearly then such a partitioning scheme facilitates the construction of the matrix of coefficients $\underline{B}$. We shall see that the analogous partitioning of the solution matrix $\underline{C}$ also facilitates the derivation of its elements. Thus, we write

$$\underline{C} = (c_{uv}), \qquad u, v = 1,\ldots,B_0+1,$$

and the "elements," or submatrices, $\underline{c}_{uv}$ have elements

$$\underline{c}_{uv} = (c_{uv}(p,q)), \qquad p,\ q = 1,\ldots,R_0+1 \ .$$

We calculate these elements in the next section.

5. CALCULATION OF C

Quite clearly, from (6), once the matrix C has been determined, the state probabilities follow immediately. As indicated earlier we exploit the underlying structure of $\underline{B}$ and the accompanying partitioning scheme. For illustrative purposes, let us take the particular case $(B_0, R_0) = (4,2)$ and $\lambda = .8$. The corresponding $\underline{B}$ matrix of coefficients is given in Table 1.

From equation (7), it is clear that

$$c_{uv}(p,q) = 0 \tag{11}$$

whenever $u < v$ and/or $p < p$. That is, the upper diagonal submatrices c_{uv} are zero and the upper diagonals of the remaining submatrices are also zero.

Let us take the elements in the first column for $(v,q) = (1,1)$.

First, from the initial condition, we have

$$c_{11}(1,1) = a_1 = 1 \ .$$

Let us proceed with the (1,1) elements of the submatrices $(u,1)$, $u = 1, \ldots, B_0+1$. We find from Severo,

$$c_{21}(1,1) = (1.6 \quad 0 \quad 0) \begin{pmatrix} 1 & & \\ - & 0 & \\ - & - & 0 \end{pmatrix} \begin{pmatrix} \delta_0(\cdot) \\ - \\ - \end{pmatrix} \tag{12}$$

where "-" indicates the presence of a "nonzero" element (which in a particular case may happen to equal zero). However because of the zeros in $\underline{b}(i,i-1)$, or in $\underline{c}_{rs}(i-1,j)$, we do not need to determine these quantities. Thus, equation (12) reduces to

$$\begin{aligned} c_{21}(1,1) &= (1.6)\ c_{11}(1,1)/\{b_{11}(1,1) - b_{22}(1,1)\} \\ &= -1.6\ . \end{aligned}$$

In general,

$$c_{u1}(1,1) = (1.6)\ c_{u-1,k}(1,1)/\{b_{11}(1,1) - b_{uu}(1,1)\}\ ;$$

or,

$$c_{u1}(1,1) = (-1)^{u-1}(1.6)^{u-1}/u!\ , \qquad u = 1,\ldots,\ B_0+1\ .$$

Let us consider the column $(v,q) = (2,1)$. Now,

$$c_{22}(1,1) = -\sum_{q=1}^{3} c_{21}(1,q) = 1.6 + 0 + 0 = 1.6\ .$$

Proceeding as before, we have

$$c_{32}(1,1) = (1.6 \quad 0 \quad 0) \begin{pmatrix} 1.6 & & \\ - & 0 & \\ - & - & 0 \end{pmatrix} \begin{pmatrix} \delta_0(\cdot) \\ - \\ - \end{pmatrix}$$

$$= (1.6)\ c_{22}(1,1)/\{b_{22}(1,1) - b_{33}(1,1)\}$$

$$= -2.56\ .$$

In general,

$$c_{u2}(1,1) = (1.6)\ c_{u-1,2}(1,1)/\{b_{22}(1,1) - b_{uu}(1,1)\}$$

$$= (-1)^u\ (1.6)^{u-1}/(u-1)!\ , \quad u = 2,\ldots,B_0+1.$$

Continuing in this manner, we can calculate all the $c_{uv}(1,1)$ elements. The essential feature to note here is that in order to determine a particular $c_{uv}(1,1)$ element, first glance at the Severo result equation (7) suggests that all previous row elements in that column of $\underset{\sim}{C}$ are necessary, whereas by exploiting the special nature of our problem we only require the previous $c_{u-1,v}(1,1)$ element.

Let us now consider the elements $c_{uv}(2,1)$. In particular, let us take the first column $(v,q) = (1,1)$. Then, from (7),

$$c_{11}(2,1) = 4c_{11}(1,1)/\{b_{11}(1,1) - b_{11}(2,2)\}$$
$$= -5.0.$$

Next,

$$c_{21}(2,1) = (0 \quad .8 \quad 0 \quad 3) \begin{pmatrix} 0 & & & \\ -5.0 & 0 & & \\ - & - & - & \\ -1.6 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_0(\cdot) \\ - \\ - \\ - \end{pmatrix}$$

$$= \{(.8)\ c_{11}(2,1) + 3c_{21}(1,1)\}/\{b_{11}(1,1) - b_{22}(2,2)\}$$

$$= 4.889.$$

In general,

$$c_{u1}(2,1)$$

$$= \{(.8)\ c_{u-1,1}(2,1) + (B_0-u+1)\ c_{u1}(1,1)\}/\{b_{11}(1,1) - b_{uu}(2,2)\} \quad ,$$

$$u = 1,\ldots,\ B_0+1 \ .$$

Similarly, we can determine that

$$c_{u2}(2,1)$$

$$= \{(.8)\ c_{u-1,2}(2,1) + (B_0-u+1)\ c_{u2}(1,1)\}/\{b_{22}(1,1) - b_{uu}(2,2)\} \ ,$$

$$u = 2,\ldots,\ B_0+1 \ .$$

Likewise we can calculate all the $c_{uv}(2,1)$ elements. Again an essential feature is that to find a particular $c_{uv}(2,1)$ we need only the two previous elements $c_{u-1,v}(2,1)$ and $c_{uu}(1,1)$ and not all the previous elements in that column.

Therefore by repeating this process, we can calculate the matrix $\underline{C}$. That is, we determine the diagonal element

$$c_{uu}(p,p) = -\sum_{v=1}^{u-1}\sum_{q=1}^{p} c_{uv}(p,q) - \sum_{q=1}^{p-1} c_{uu}(p,q)$$

Once each diagonal element is calculated, the other elements in that same column $(v,q) = (u,p)$ can be determined according to

$$c_{uv}(p,q)$$

$$= \{\lambda(R_0-p+1)\ c_{u-1,v}(p,q)+(B_0-u+1)\ c_{uv}(p-1,q)\}/\{b_{vv}(q,q)-b_{uu}(p,p)\} \tag{13}$$

for $u = v,\ldots, B_0+1$, $v = 1,\ldots, B_0+1$, $p = q,\ldots,R_0+1$, $q = 1,\ldots,R_0+1$, and where equation (11) applies when applicable in (13).

For the computer user what would be a potential storage problem if all the previous $c_{uv}(p,q)$ elements were needed, does not cause any undue concern since only the two appropriate values need be retained. This assumes the program has been set up to print out the values of $\underline{C}$ as they are determined and that a running sum of the row elements is retained (for the calculation of subsequent diagonal elements; see equation (7)).

The complete matrix $\underline{C}$ for our example is shown in Table 2.

6. THE STATE PROBABILITIES

Once the elements of $\underline{C}$ have been calculated, the state probabilities $p_{b,r}(t)$ follow readily from (3) and (6). Thus, for example,

$$\Pr\{B(t) = 2,\ R(t) = 1\} = p_{2,1}(t)$$

$$= -2.311e^{-5.6t} + 1.6e^{-4.8t} + 5.511e^{-4.6t} - 4.089e^{-3.8t} - 3.200e^{-3.66} + 2.489e^{-2.8t} ,$$

and

Pr {Red forces lose no units}

$$= \sum_{b=0}^{4} \Pr\{B(t) = b,\ R(t) = 2\}$$

$$= 0.192e^{-5.6t} + 0.376e^{-4.6t} + 0.142e^{-3.6t} + 0.263e^{-2.6t} + 0.027 .$$

In general,

$$\Pr\{B(t) = b,\ R(t) = r\}$$

$$= \sum_{v=1}^{B_0-b} \sum_{q=1}^{R_0+1} c_{B_0-b+1,v}(R_0-r+1,q)\ \exp(tb_{vv}(q,q))$$

$$+ \sum_{q=1}^{R_0-r+1} c_{B_0-b+1,B_0-b+1}(R_0-r+1,q)\ \exp(tb_{B_0-b+1,B_0-b+1}(q,q)) .$$

Or, more simply, in terms of the k notation used before partitioning,

$$p_{b,r}(t) \equiv y_k(t) = \sum_{j=1}^{k} c(k,j)\, \exp(b_j t) \ .$$

A knowledge of the state probabilities permits the derivation of other quantities of interest, for example, the expected force sizes.

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TABLE 1

Matrix of coefficients (of transition generators) $\underline{B}$ when $B_0 = 4$, $R_0 = 2$, $\lambda = .8$

$$
B = \begin{bmatrix}
-5.6 & & & & & & & & & & & & & & \\
4 & -4.8 & & & & & & & & & & & & & \\
 & 4 & 0 & & & & & & & & & & & & \\
1.6 & & & -4.6 & & & & & & & & & & & \\
 & .8 & & 3 & -3.8 & & & & & & & & & & \\
 & & 0 & & 3 & 0 & & & & & & & & & \\
 & & & 1.6 & & * & -3.6 & & & & & & & & \\
 & & & & .8 & & 2 & -2.8 & & & & & & & \\
 & & & & & 0 & & 2 & 0 & & & & & & \\
 & & & & & & 1.6 & & & -2.6 & & & & & \\
 & & & & & & & .8 & & 1 & -1.8 & & & & \\
 & & & & & & & & 0 & & 1 & 0 & & & \\
 & & & & & & & & & 1.6 & & & 0 & & \\
 & & & & & & & & & & .8 & & 0 & 0 & \\
 & & & & & & & & & & & 0 & & 0 & 0
\end{bmatrix}
\quad
\begin{matrix}
(b,r) \\
4\ 2 \\ 4\ 1 \\ 4\ 0 \\ 3\ 2 \\ 3\ 1 \\ 3\ 0 \\ 2\ 2 \\ 2\ 1 \\ 2\ 0 \\ 1\ 2 \\ 1\ 1 \\ 1\ 0 \\ 0\ 2 \\ 0\ 1 \\ 0\ 0
\end{matrix}
$$

TABLE 2

Coefficients of "solution" matrix $\underline{C}$ when $B_0 = 4$, $R_0 = 2$, $\lambda = .8$

$$
\underline{C} = \left[\begin{array}{ccccccccccccccc}
1 & & & & & & & & & & & & & & \\
-5.0 & 5.0 & & & & & & & & & & & & & \\
3.571 & -4.167 & .596 & & & & & & & & & & & & \\
-1.6 & 0 & 0 & 1.6 & & & & & & & & & & & \\
4.889 & -4.0 & 0 & -6.0 & 5.111 & & & & & & & & & & \\
-2.619 & 2.50 & 0 & 3.913 & -4.035 & .241 & & & & & & & & & \\
1.28 & 0 & 0 & -2.56 & 0 & 0 & 1.28 & & & & & & & & \\
-2.311 & 1.60 & 0 & 5.511 & -4.089 & 0 & -3.20 & 2.489 & & & & & & & \\
.825 & -.667 & 0 & -2.396 & 2.152 & 0 & 1.778 & -1.778 & .086 & & & & & & \\
-.683 & 0 & 0 & 2.048 & 0 & 0 & -2.048 & 0 & 0 & .683 & & & & & \\
.666 & -.427 & 0 & -2.306 & 1.636 & 0 & 2.560 & -1.991 & 0 & -.854 & .716 & & & & \\
-.119 & .089 & 0 & .501 & -.431 & 0 & -.711 & .711 & 0 & .329 & -.398 & .029 & & & \\
.195 & 0 & 0 & -.712 & 0 & 0 & .910 & 0 & 0 & -.420 & 0 & 0 & .027 & & \\
-.095 & .071 & 0 & .401 & -3.443 & 0 & -0.566 & .569 & 0 & 2.627 & -.318 & 0 & 0 & .754 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

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